

GROUP ALGEBRAS WHOSE INVOLUTORY UNITS COMMUTE

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Dedicated to the memory of Professor I. I. Khripta

ABSTRACT. Let K be a field of characteristic 2 and G a nonabelian locally finite 2-group. Let $V(KG)$ be the group of units with augmentation 1 in the group algebra KG . An explicit list of groups is given, and it is proved that all involutions in $V(KG)$ commute with each other if and only if G is isomorphic to one of the groups on this list. In particular, this property depends only on G and not at all on K .

Introduction.

Let KG be the group algebra of a locally finite p -group G over a field K of characteristic p . Then the normalized unit group

$$V(KG) = \left\{ \sum_{g \in G} \alpha_g g \in KG \mid \sum_{g \in G} \alpha_g = 1 \right\}$$

is a locally finite p -group.

An interesting way to study $V(KG)$ is to construct embeddings of important groups into it. D.B. Coleman and D.S. Passman [4] have proved that if G is non-abelian then the wreath product of two cyclic groups of order p is involved in $V(KG)$. Larger wreath products have been constructed by A. Mann and A. Shalev in [7], [8], [9] and [10]. In this paper we answer the question of when dihedral groups cannot be embedded into $V(KG)$. Clearly, only the case $p = 2$ has to be considered. At the same time we obtain the list of the locally finite 2-groups G such that $V(KG)$ does not contain a subgroup isomorphic to a wreath product of two groups (for the case of odd p see [1]).

We remark, that in the present work the method of proof developed in [3] plays an essential role.

Let C_{2^n} , C_{2^∞} and Q_8 be the cyclic group of order 2^n , the quasicyclic group of type 2^∞ and the quaternion group of order 8, respectively. An *involution* is a group element of order 2. For any $a, b \in G$ we denote $[a, b] = a^{-1}b^{-1}ab$, $a^b = b^{-1}ab$.

Our main result is the following:

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Theorem. *Let K be a field of characteristic 2, and G a locally finite nonabelian 2-group. Then all involutions of $V(KG)$ commute if and only if G is one of the following groups:*

- (i) $S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}} \rangle$ with $n, m \geq 2$, or Q_8 ;
- (ii) $Q_8 \times C_{2^n}$ or $Q_8 \times C_{2^\infty}$;
- (iii) the semidirect product of the cyclic group $\langle d \mid d^{2^n} = 1 \rangle$ with the quaternion group $\langle a, b \mid a^4 = 1, a^2 = b^2 = [a, b] \rangle$ such that $[a, d] = d^{2^{n-1}}$ and $[b, d] = 1$;
- (iv) $H_{32} = \langle x, y, u \mid x^4 = y^4 = 1, x^2 = [y, x], y^2 = u^2 = [u, x], x^2y^2 = [u, y] \rangle$.

For a 2-group G we denote by $\Omega(G)$ the subgroup generated by all elements of order 2 of G . We write $\Omega(V) = \Omega(V(KG))$. Clearly, $\Omega(G)$ is a normal subgroup of G . As usually, $\exp(G)$, $C_G(\langle a, b \rangle)$ denote the exponent of a group G and the centralizer of the subgroup $\langle a, b \rangle$ in G , respectively. Let

$$\mathcal{L}(KG) = \langle xy - yx \mid x, y \in KG \rangle$$

be the commutator subspace of KG .

For an element g of a finite order $|g|$ in a group G , let \bar{g} denote the sum (in KG) of the distinct powers of g :

$$\bar{g} = \sum_{i=0}^{|g|-1} g^i.$$

For an arbitrary element $x = \sum_{g \in G} \alpha_g g \in KG$ we put $\chi(x) = \sum_{g \in G} \alpha_g \in K$.

It will be convenient to have a short temporary name for the locally finite 2-groups G such that all elements of order 2 of $V(KG)$ form an abelian subgroup. Let us call the groups G with this property *good*.

1. Preliminary results.

Lemma 1. *Let G be a finite nonabelian good group. Then all involutions of G are central, $G' \subseteq \Omega(G)$ and either G is the quaternion group of order 8 or $\Omega(G)$ is a direct product of two cyclic groups.*

Proof. Let G be a nonabelian good group. Clearly $\Omega(V) \cap G = \Omega(G)$ is a normal abelian subgroup of G .

Suppose that $|\Omega(G)| = 2$. Then by Theorem 3.8.2 in [5] we have that

$$G = \langle a, b \mid a^{2^m} = 1, b^2 = a^{2^{m-1}}, a^b = a^{-1} \rangle$$

with $m \geq 2$ is a generalized quaternion 2-group. If $|G| > 8$, then we choose $c \in \langle a \rangle$ of order 8. Then $1 + (1 + c^2)(c + b)$ and $1 + (1 + c^2)(c + cb)$ are noncommuting involutions of $V(KG)$, which is impossible. Therefore, G is the quaternion group of order 8.

Let $|\Omega(G)| > 2$. The normal subgroup $\Omega(G)$ contains a central element a in G and $x = 1 + (a + 1)g$ is an involution for any $g \in G$. If $b \in \Omega(G)$, then $bx = xb$ and $(a + 1)(1 + [b, g]) = 0$. It implies that $[g, b] = 1$ or $[g, b] = a$. Let $[g, b] = a$ and $|g| = 2^t$. Then $z = 1 + (g + 1)\bar{g}^2$ is an involution and $zb = bz$. From this we get that $a \in \langle g^2 \rangle$ and since $a \in \Omega(G)$ we obtain that $a = g^{2^{t-1}}$. Clearly, if $t = 2$, then $\langle b, g \rangle$ is the dihedral group of order 8, which is impossible. If $t > 2$, then $x = 1 + g(1 + g^{2^{t-2}})(1 + b)$ is an involution which does not commute with b , which

is a contradiction. Therefore, $[g, b] = 1$ and $\Omega(G)$ is a central subgroup of G . Let a and b be arbitrary elements of $\Omega(G)$, $g, h \in G$ and $[g, h] \neq 1$. Then $x = 1 + (a + 1)g$ and $y = 1 + (b + 1)h$ are involutions and $[x, y] = 1$. From this we conclude that the commutator subgroup G' is a subgroup of $\langle a, b \rangle$ in $\Omega(G)$ and $\exp(G') = 2$.

Now, let $|\Omega(G)| \geq 8$ and let a, b, c be linearly independent elements of $\Omega(G)$. Then by the above reasoning we have $[g, h] \in \langle a, b \rangle \cap \langle a, c \rangle \cap \langle b, c \rangle = 1$, which is impossible. Therefore, $|\Omega(G)| = 4$ and $G' \subseteq \Omega(G)$. \square

Lemma 2. *A two-generator finite nonabelian group is good if and only if it is either the quaternion group of order 8 or*

$$S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}} \rangle \quad (1)$$

with $n, m \geq 2$.

Proof. Suppose that G is not the quaternion group of order 8. By Lemma 1 we have that $\Omega(G)$ is a direct product of two cyclic groups and $G' \subseteq \Omega(G)$. Then the Frattini subgroup $\Phi(G) = \{g^2 \mid g \in G\}$ is central and by Theorem 3.3.15 in [5], $|G/\Phi(G)| = 4$. Since $\Phi(G)$ is a subgroup of the centre $\zeta(G)$ and the factor group $G/\zeta(G)$ can not be cyclic, this implies $\Phi(G) = \zeta(G)$.

It is easy to verify that a two-generator good group G is metacyclic. Indeed, every maximal subgroup M of G is abelian and normal in G , because $\Phi(G) = \zeta(G) \subset M$ and $|M/\zeta(G)| = 2$. Clearly, $\Omega(M) \subseteq \Omega(G)$ and in case $|\Omega(M)| = 2$ the subgroup M is cyclic and we conclude that G is metacyclic.

Now let $|\Omega(M)| = 4$ for every maximal subgroup M of G . Then G and M are two-generator groups. It is easy to see that all such groups of order 16 are metacyclic. If $|G| = 2^n$ ($n \geq 5$) then G and all maximal subgroups of G are two generator groups and by Theorem 3.11.13 in [5] G is metacyclic too. Since $G' \subseteq \Omega(G)$ by Lemma 1, it follows from Theorem 3.11.2 in [5], G is defined by

$$G = \langle a, b \mid a^{2^n} = 1, b^{2^m} = a^{2^l}, a^b = a^{1+2^{n-1}} \rangle$$

with $n, m \geq 2$.

Suppose that the generators a and b are choosen with minimal order of b . We want to show that $b^{2^m} = 1$. Suppose by contradiction that the order of b is bigger than 2^m . Then $m \leq l$. Indeed, if $m > l$ then $b^{2^{m-l}} \in \zeta(G)$ and $(ab^{-2^{m-l}})^{2^l} = 1$, so we may take generators $a_1 = b$ and $b_1 = ab^{-2^{m-l}}$, but the order of b_1 is less than 2^m , a contradiction with our choice of generators. Thus, $m \leq l$. If $m < l$, then $a^{2^{l-m}}$ is central and, taking $a_1 = a, b_1 = a^{-2^{l-m}}b$, we have that $(a^{-2^{l-m}}b)^{2^m} = 1$ and the order of b_1 is less then the order of b , which is not allowed. Now, the case $l = m > 1$ is also impossible, since we can take generators $a_1 = a, b_1 = a^{-1}b$ with $b_1^{2^m} = 1$. In the last case, when $l = m = 1$, we have $n \geq 3$ as G is not quaternion of order 8. Hence, $a^{2^{n-2}} \in \zeta(G)$ and $a^{2^{n-2}-2}ab$ is a non-central element of order 2, which gives the final contradiction. \square

Lemma 3. (E. A. O'Brien, see Lemma 4.1 in [3]) *The groups H of order dividing 128 in which $\Phi(H)$ and $\Omega(H)$ are equal, central, and of order 4, are precisely the following: $C_4 \times C_4$, $C_4 \rtimes C_4$, $C_4 \rtimes Q_8$, $Q_8 \times C_4$, $Q_8 \times Q_8$, the central product of*

the group $S_{2,2} = \langle a, b \mid a^4 = b^4 = 1, a^2 = [b, a] \rangle$ with a quaternion group of order 8, the nontrivial element common to the two central factors being a^2b^2 ,

$$\begin{aligned} H_{245} = \langle x, y, u, v \mid & x^4 = y^4 = [v, u] = 1, \\ & x^2 = v^2 = [y, x] = [v, y], \\ & y^2 = u^2 = [u, x], \\ & x^2y^2 = [u, y] = [v, x] \rangle \end{aligned}$$

and the groups named in parts (iii), (iv) of Theorem.

□

The group H_{245} , is one of the two Suzuki 2-groups of order 64.

2. Proof of the ‘only if’ part of the theorem.

Let G be a finite nonabelian good group, so, $\Omega(V)$ is abelian.

By Lemma 1 all involutions of G are central, $G' \subseteq \Omega(G)$ and $\Omega(G)$ is either a group of order 2 or a direct product of two cyclic groups. Clearly, $\Phi(G) \subseteq \zeta(G)$ and if $|\Omega(G)| = 2$ then by Lemma 1 G is a quaternion group of order 8. Thus, we can suppose that $|\Omega(G)| = 4$.

First let $\Phi(G)$ be cyclic. Since all involutions are central, by Theorem 2 in [2] G is the direct product of a group of order 2 and the generalized quaternion group of order 2^{n+1} . By Lemma 1 G is a Hamiltonian 2-group of order 16.

We may suppose that $\Phi(G)$ is the direct product of two cyclic groups. Let the exponent of G be 4. Then $\Phi(G) = \Omega(G)$ and by a result of N. Blackburn (Theorem VIII.5.4 in [6]), $|G| \leq |\Omega(G)|^3$. Therefore, the order of G divides 64. Then by O’Brien’s Lemma, G is precisely one of the following types: $C_4 \rtimes C_4$, $C_4 \rtimes Q_8$, $Q_8 \times C_4$, the groups named in parts (iii), (iv) of Theorem and $Q_8 \times Q_8$, H_{245} , the central product of Q_8 and $S_{2,2}$ with common a^2b^2 .

Now we shall find noncommuting involutions z_1, z_2 in $V(KG)$ if G is one of the last three groups listed above.

Let G be the central product of the group

$$S_{2,2} = \langle a, b \mid a^4 = b^4 = 1, a^2 = [b, a] \rangle$$

with the quaternion group of order 8, the nontrivial element common to the two central factors being a^2b^2 . Then

$$\begin{aligned} G \cong \langle a, b, d, f \mid & a^4 = d^4 = 1, b^2 = a^2 = [a, b], f^2 = d^2 = [d, f], \\ & [a, d] = [b, d] = [b, f] = 1, [a, f] = a^2 \rangle \end{aligned}$$

and we put $z_1 = 1 + d^2a + b + a^3d + bd + f + abf + df + abdf$ and $z_2 = 1 + b(1 + d^2)$. If $G \cong H_{245}$ then

$$\begin{aligned} H_{245} \cong \langle a, b, d, f \mid & a^4 = b^4 = 1, b^2 = d^2 = a^2, [a, b] = 1, \\ & [a, d] = [b, f] = [d, f] = b^2, [b, d] = a^2, [a, f] = a^2b^2 \rangle \end{aligned}$$

and put $z_1 = 1 + a + ab + d + a^2bd + f + bf + ab^2df + a^3b^3df$ and $z_2 = 1 + (b + b^{-1})$. Now, let G be a direct product of two quaternion groups $\langle a, b \rangle$ and $\langle c, d \rangle$ of order 8. Then we put $z_1 = 1 + a + bc^2 + c + abc + a^2d + abd + acd + bcd$ and $z_2 = 1 + b(1 + c^2)$.

It is easy to verify that in all three cases $z_1^2 = z_2^2 = 1$, $z_1 z_2 \neq z_2 z_1$.

Now, let the exponent of G be greater than 4. Using Lemma 2, we conclude that G contains a two-generator nonabelian subgroup H which is either Q_8 or $S_{n,m}$.

We wish to prove that if $\exp(G) > 4$ and $G = H \cdot C_G(H)$ for every two-generator nonabelian subgroup H , then

$$G = Q_8 \times \langle d \mid d^{2^n} = 1, n > 1 \rangle.$$

First, let $H = Q_8 = \langle a, b \rangle$ be a quaternion subgroup of order 8 of G . Then $G = Q_8 \cdot C_G(Q_8)$ and $C_G(Q_8)$ does not contain an element c of order 4 with the property $c^2 = a^2$, because ac would be a noncentral involution of G , which is impossible. If $C_G(Q_8)$ is abelian and $|\Omega(C_G(Q_8))| = 4$ then $C_G(Q_8)$ is the direct product of $\langle a^2 \rangle$ and $\langle d \mid d^{2^n} = 1 \rangle$, with $n > 1$, and $G = Q_8 \times \langle d \rangle$.

We can suppose that $C_G(Q_8)$ is nonabelian and does not contain an element u such that $u^2 = a^2$. Since $\exp(C_G(Q_8)) > 4$, there always exists a subgroup

$$S_{n,m} = \langle c, d \mid c^{2^n} = d^{2^m} = 1, c^d = c^{1+2^{n-1}} \rangle$$

of $C_G(Q_8)$ which is of exponent greater than 4. Then $\zeta(S_{n,m}) = \langle c^2, d^2 \rangle$ and since $\exp(S_{n,m}) > 4$, one of the generators c or d has order greater than 4. Therefore, any $u \in \Omega(\zeta(S_{n,m}))$ is the square of one of the elements from $S_{n,m}$. Thus, since we assume that $C_G(Q_8)$ does not contain an element of order 4 whose square is a^2 , we get that $S_{n,m} \cap Q_8 = 1$ and $S_{n,m} \times Q_8$ is a subgroup of G with the property $|\Omega(S_{n,m} \times Q_8)| = 8$, which is impossible.

Now let $H = S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}} \rangle$ with $n, m \geq 2$ be a subgroup of G . Then $G = S_{n,m} \cdot C_G(S_{n,m})$. Since $|\Omega(G)| = 4$, we can choose $d \in C_G(S_{n,m})$ such that $d \notin S_{n,m}$ but $d^2 \in S_{n,m}$. Then $d^2 = a^{2^i} b^{2^j}$. If i or j is even then $d^{-1} a^i b^j \in \Omega(G) = \Omega(S_{n,m})$ and $d \in S_{n,m}$, which is impossible. If i and j are odd and $n > 2$ then $d^{-1} a^{i+2^{n-2}} b^j \in \Omega(G) = \Omega(S_{n,m})$ which is a contradiction. Therefore, $n = 2$ and $\langle a^i, d^{-1} b^j \rangle$ is a quaternion subgroup and by assumption $G = Q_8 \cdot C_G(Q_8)$. We obtained the previous case.

It is easy to check that if the commutator subgroup G' is of order 2, then $G = H \cdot C_G(H)$ for every two-generator nonabelian subgroup of G . Indeed, the equalities $G' = H' = \langle c \rangle$ imply that $H = \langle a, b \rangle$ is a normal subgroup of G . Let $[a, b] = c$, $[a, g] = c^k$, $[b, g] = c^l$, where $0 \leq k, l \leq 1$, $g \in G$. Then at least one of the elements g , ag , bg , abg belongs to $C_G(H)$ and $g \in H \cdot C_G(H)$. Therefore, $G = H \cdot C_G(H)$.

It follows that we can suppose that $\exp(G) > 4$, the commutator subgroup $G' = \Omega(G)$ has order 4 and G contains a two-generator nonabelian subgroup L such that $G \neq L \cdot C_G(L)$.

Let $L = \langle b, d \mid [b, d] \neq 1 \rangle$ and $a \in G \setminus (L \cdot C_G(L))$. Then $[a, d]$ or $[b, a]$ is not equal to $[b, d]$. Indeed, in the contrary case, from $[a, b] = [d, b]$ and $[a, d] = [b, d]$ we get $bda \in C_G(L)$ and $a \in L \cdot C_G(L)$ which is impossible.

Now we want to prove that we can choose $a \in G \setminus (L \cdot C_G(L))$ and $b, d \in L$ such that $\langle b, d \rangle = L$, $[a, b] = 1$ with the following property:

$$[a, d] \neq [b, d], [a, d] \neq 1, [b, d] \neq 1. \quad (2)$$

If $[a, d] = 1$ then we can put $a' = a, b' = d$ and $d' = b$.

We consider the following cases:

Case 1. Let $[b, d] = [b, a] \neq [a, d]$. Then $[b, ad] = 1$ and we put $a' = ad$, $b' = b$ and $d' = d$. If $[a', d'] = 1$ then $ad \in C_G(L)$ which implies that $a \in L \cdot C_G(L)$, a contradiction.

Case 2. Let $[b, d] = [a, d] \neq [a, b]$. Then $[ab, d] = 1$ and put $a' = ab$, $b' = d$ and $d' = b$. If $[a', d'] = 1$ then $ab \in C_G(L)$ which gives a contradiction again.

Case 3. Let $[a, b] \neq [a, d] \neq [b, d] \neq [a, b]$. Suppose that all these commutators are not trivial. Since $|\Omega(G)| = 4$, one of these commutators equals to the product of two others and

$$[ab, bd] = [a, b] \cdot [b, d] \cdot [a, d] = 1.$$

Put $a' = ab$, $b' = bd$ and $d' = d$.

In all what follows we suppose that $L = \langle b, d \rangle$ and $a \in G \setminus (L \cdot C_G(L))$ such that $[a, b] = 1$ and (2) is satisfied.

It is easy to see that if $\langle a, b \rangle = \langle u \rangle$ is cyclic then from $[a, d] \neq 1 \neq [b, d]$ we have $a = u^{2k+1}$ and $b = u^{2t+1}$ for some $k, t \in \mathbb{N}$, because the squares of all elements in G are central. Then $ab \in C_G(L) \subseteq L \cdot C_G(L)$ and $a \in L \cdot C_G(L)$, which is impossible. Therefore, $\langle a, b \rangle$ is not cyclic.

Consider, $W = \langle a, b, d \rangle$. Then the commutator subgroup of W has order 4 and

$$H = \langle a, b \rangle = \langle a_1 \rangle \times \langle b_1 \rangle.$$

Clearly $W = \langle a_1, b_1, d \rangle$ and $|W'| = 4$. It is easy to see that a_1 and b_1 can be choosen such that condition (2) is satisfied and $\langle a_1 \rangle \cap \langle b_1 \rangle = 1$.

Let $a, b, d \in G$ with the property (2), $[a, b] = 1$ and $\langle a \rangle \cap \langle b \rangle = 1$. Put $H = \langle a \mid a^{2^n} = 1 \rangle \times \langle b \mid b^{2^m} = 1 \rangle$ and $W = \langle a, b, d \rangle$. Then

$$G' = \Omega(G) = \Omega(H) = \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle$$

and H is a normal subgroup of G .

First, we shall prove that $g^2 \in H$ for every $g \in G \setminus H$. There exists $c = g^{2^{k-1}}$ such that $c \notin H$ and $c^2 \in H$. If $k > 1$, then $c \in \Phi(G) \subset \zeta(G)$ and we obtain that $c^2 \in \zeta(G) \cap H$ and $c^2 = a^{2i}b^{2j}$. Thus, $(c^{-1}a^ib^j)^2 = 1$ and $c^{-1}a^ib^j \in \Omega(G) = \Omega(H)$. It implies that $c \in H$, which is impossible. Therefore, $k = 1$ and $g^2 = a^{2i}b^{2j}$ for some i and j , and we have shown that $g^2 \in H$ for all $g \notin H$.

We have $[g, a^ib^j] = a^{s2^{n-1}}b^{r2^{m-1}}$ for some $r, s \in \{0, 1\}$. It is easy to see that the case $n > 2$ and $m > 2$ is impossible. Indeed, if $n > 2$ and $m > 2$ then $a^{i+s2^{n-2}} \cdot b^{j+r2^{m-2}} \cdot g^{-1} \in \Omega(G) = \Omega(H)$ and $g \in H$, which is a contradiction.

Since $\exp(G) > 4$ and for any $g \in G$, $g^2 = a^{2i}b^{2j}$ for some i, j it follows that $\exp(H) > 4$. Thus we may suppose that $n > 2$ and b is an element of order 4.

Now we describe the group $W = \langle a, b, d \rangle$ and we distinguish a number of cases according to the form of the element d^2 .

Case 1. Let $d^2 = a^{2i}$. Since $d \notin H$ and a^id^{-1} is not of order 2, we have that $i = 2k + 1$ is odd. Then as $(a^{i+2^{n-2}}d^{-1})^2 \neq 1$ we have

$$(a^{i+2^{n-2}}d^{-1})^2 = [a^i, d] \cdot a^{2^{n-1}} = [a^{2k+1}, d] \cdot a^{2^{n-1}} = [a, d] \cdot a^{2^{n-1}} \neq 1.$$

It follows that $[a, d] \neq a^{2^{n-1}}$ and $[a, d] = a^{s2^{n-1}}b^2$, $s \in \{0, 1\}$, and by property (2) $[b, d] = a^{(1+s)2^{n-1}}b^2$ or $[b, d] = a^{2^{n-1}}$. If $[b, d] = a^{2^{n-1}}$ then $a^{i+(1+s)2^{n-2}}bd^{-1}$ has order 2 and does not belong to $\Omega(H)$ and this case is impossible.

Let $[a, d] = a^{s2^{n-1}}b^2$ and $[b, d] = a^{(1+s)2^{n-1}}b^2$. Then

$$W = \langle a, b, d \rangle = \langle a^{i+2^{n-2}}d^{-1}, a^{(1+s)2^{n-2}}b, ab \rangle$$

and $\langle a^{i+2^{n-2}}d^{-1}, a^{(1+s)2^{n-2}}b \rangle$ is the quaternion subgroup of order 8. Moreover, ab has order 2^n and $[a^{i+2^{n-2}}d^{-1}, ab] = a^{2^{n-1}} = (ab)^{2^{n-1}}$. This shows that W satisfies (iii) of the Theorem. Observe that $\langle a^{(1+s)2^{n-2}}b, ab \rangle = H$.

Case 2. Let $d^2 = b^2$. As before,

$$(a^{2^{n-2}}bd^{-1})^2 = a^{2^{n-1}}[b, d] \neq 1.$$

Therefore, $[b, d] \neq a^{2^{n-1}}$, and we obtain $[b, d] = a^{s2^{n-1}} \cdot b^2$, and by property (2) $[a, d] = a^{(1+sr)2^{n-1}} \cdot b^{2r}$, where $r, s \in \{0, 1\}$. It is easy to see that $\langle bd^{-1}, a^{s2^{n-2}}b \rangle$ is the quaternion group of order 8 and $W = \langle bd^{-1}, a^{s2^{n-2}}b, ab^r \rangle$ is defined as in case 1 and satisfies condition (iii) of the Theorem. Moreover, $\langle a^{s2^{n-2}}b, ab^r \rangle = H$.

Case 3. Let $d^2 = a^{2i}b^2$. If i is even then $a^i \in \zeta(G)$ and $(da^{-i})^2 = b^2$. Then $W = \langle a, b, da^{-i} \rangle$ and if we replace d by $d' = da^{-i}$ we obtain $d'^2 = b^2$ which is case 2.

Let now $d^2 = a^{2i}b^2$ and i is odd. If $[a, d] = a^{s2^{n-1}} \cdot b^2$, then $a^{i+s2^{n-2}} \cdot d^{-1} \in \Omega(H)$ and $d \in H$, which is impossible. Therefore, $[a, d] = a^{2^{n-1}}$ and $(a^{i+2^{n-2}} \cdot d^{-1})^2 = b^2$.

If we replace d by $d' = a^{i+2^{n-2}}d^{-1}$, we obtain $W = \langle a, b, d' \rangle$ and $(d')^2 = b^2$, which is case 2 and we have that W satisfies (iii) of the Theorem.

This proves that the subgroup W has a generator system u, v, w such that

$$W = \langle w, u, v \mid w^4 = 1, w^2 = v^2, v^w = v^{-1}, u^{2^n} = 1, u^w = u^{1+2^{n-1}}, [u, v] = 1 \rangle,$$

with $n > 2$ and $H = \langle u, v \rangle$.

Suppose that there exists $g \in G \setminus W$. Clearly, $G' \subseteq W$ and W is normal in G . Above we proved that the squares of all elements of G outside W belong to H and they are central in W . Therefore, by the above argument we get that $g^2 = u^{2s}v^{2t}$ for every $g \in G \setminus W$, where $t \in \{0, 1\}$, $s \in \mathbb{N}$. It is easy to see that

$$(g^{-1}u^s)^2 = [g, u^s]g^{-2}u^{2s} = [g, u^s]v^{-2t}.$$

If $(g^{-1}u^s)^2 = 1$ then $g^{-1}u^s \in \Omega(W) \subseteq W$ and $g \in W$, which is impossible. Clearly, the order of elements $[g, u^s]$ and v^{-2t} divide 2 and $g^{-1}u^s$ is an element of order 4. Then $M = \langle g^{-1}u^s, v, w, u^{2^{n-2}} \rangle$ is a subgroup of exponent 4 and $\Omega(M) = \Omega(G)$. Clearly, $M/\Omega(M)$ is an elementary abelian 2-subgroup of order 16. Therefore, M is a group with four generators. By O'Brien's Lemma, M is isomorphic either to $Q_8 \times Q_8$ or to H_{245} or to the central product $S_{2,2}$ and Q_8 with common a^2b^2 . It is impossible, because the centres of these groups have exponent 2 but in M there exists a central element $u^{2^{n-2}}$ of order 4. It completes the description of finite good groups.

Now we suppose that G is an infinite good group. We shall prove that G is the direct product of the quaternion group of order 8 and the quasicyclic 2-group.

It is easy to see that if G has exponent 4 then G is finite. Indeed, if G is abelian then its finiteness follows from the first Prüfer's Theorem and the condition $|\Omega(G)| \leq 4$. If G is nonabelian then take an ascending chain $G_1 \subset G_2 \subset \dots$ of

finite subgroups of G . It follows from the description of finite good groups given above that the above chain is finite, that is $G_n = G_{n+1} = \dots$ for some $n \in \mathbb{N}$. Since G is locally finite, $G = G_n$, a contradiction.

Thus, we may suppose that $\exp(G) > 4$. We show now that $\zeta(G)$ contains a divisible subgroup. Let $\mathcal{U}_2(G) = \langle g \in G \mid g^4 = 1 \rangle$ and $\Omega_2(G) = \langle g^4 \mid g \in G \rangle$. Then $(gh)^4 = g^4 h^4$ for all $g, h \in G$ and the map $g \rightarrow g^4$ is a group homomorphism of G onto $\Omega_2(G)$ with kernel $\mathcal{U}_2(G)$. Since $\Omega_2(G)$ has exponent 4, by the above paragraph, $\Omega_2(G)$ is finite. If $\zeta(G)$ does not contain a divisible group, $\zeta(G)$ is finite. Hence, $\mathcal{U}_2(G) \subseteq \zeta(G)$ is finite too which implies the finiteness of G , a contradiction.

We have that $\zeta(G) = R \times P$, where $1 \neq P$ is divisible and R does not contain a divisible subgroup. Observe now that $R \neq 1$ is cyclic, P is quasicyclic and for every noncentral $g \in G$ there exists a noncentral element $g_1 \in G$ such that

$$g_1 \equiv g \pmod{P} \text{ and } \langle g_1^2 \rangle = R. \quad (3)$$

Indeed, we have that $g^2 = cc'$, where $c \in R, c' \in P$. Taking $g_1 = gd^{-1}$ with $d^2 = c'$ we get $g_1^2 = c$, $g_1 \equiv g \pmod{P}$. As g_1 is noncentral, $c \neq 1$. It follows that $R \neq 1$ and since $|\Omega(G)| = 4$, R is cyclic and P is a quasicyclic. Let $R = \langle z \mid z^{2^n} = 1 \rangle$ with $n \geq 1$ and $P = \langle c_1, c_2, \dots, c_k, \dots \mid c_1^2 = 1, c_{k+1}^2 = c_k \rangle$ with $k = 1, 2, \dots$. If $c = z^l$ with even l , then $g_1 z^{-\frac{l}{2}}$ is a noncentral element of order 2, which is impossible. Hence, l is odd and $\langle g_1^2 \rangle = R$ as desired in (3).

Next we observe that R has order 2 (that is $n = 1$). Let g and t be two noncommuting elements in G . We have that $[g, t] = z^{2^{n-1}i} c_1^j$ ($i, j \in \{0, 1\}$) and by (3) we can suppose that $g^2 = z^{i_1}, t^2 = z^{i_2}$ with $i_1 \equiv i_2 \equiv 1 \pmod{2}$. Choose m_1 and m_2 , such that $i_1 m_1 \equiv i_2 m_2 \equiv 1 \pmod{2^n}$. If $n > 1$, then $x = g^{m_1} t^{-m_2} z^{i_2 m_2} c_2^j$ has order 2 and $[x, t] \neq 1$, a contradiction. Thus R has order 2 and $g^2 = t^2 = z$. If $i = 0$ then $gt^{-1} c_2^j$ is a noncentral element of order 2, a contradiction. Therefore, $[g, t] = z c_1^j$ and $Q = \langle g c_2^j, t c_2^j \rangle$ is isomorphic to the quaternion group of order 8.

We show now that $G = Q \times P$ and it will complete the proof of necessity of the Theorem. Fix a noncentral element x of G . There exists an element $c \in P$ such that $(xc)^2 = (gc_2^j)^2$. Indeed, if $j = 0$ this follows from (3). In case $j = 1$ by (3) take xc' such that $(xc')^2 = z$. Then $(xc' c_2)^2 = z c_1$ as we need.

It is enough to show that $x_1 = xc \in W = Q \times \langle c_3 \rangle$. Suppose that $x_1 \notin W$. Then $W_1 = \langle x_1, W \rangle$ has order 128. Since W_1 contains an element of order 8, by the description of the finite good groups given above, W_1 is one of the finite groups given in (ii) and (iii) of the Theorem.

If $W_1 = Q_1 \times \langle d \mid d^{16} = 1 \rangle$, where Q_1 is isomorphic to the quaternion group of order 8, then $W = Q_1 \times \langle d^2 \rangle$ and as $x_1 \notin W$, $x_1 = vd$ for some $v \in W$. Hence, $x_1^8 = d^8 \neq 1$, contradicting the fact that $x_1^4 = 1$.

If W_1 is given by (iii) of the Theorem, then $W = \langle a, b \rangle \times \langle d^2 \rangle$ and $x_1 = vd$ for some $v \in W$. Then we have again that $x_1^8 = d^8 \neq 1$ which is impossible.

Thus, $x_1 \in W$ and, consequently, $G \cong Q \times C_{2^\infty}$.

3. Proof of the 'if' part of the theorem. We shall prove that $\Omega(V) = 1 + \mathfrak{I}(\Omega(G))$, where $\mathfrak{I}(\Omega(G))$ is the ideal generated by all elements of form $g - 1$ with $g \in \Omega(G)$. If $|\Omega(G)| = 2$ then $\mathfrak{I}^2(\Omega(G)) = 0$ and $1 + \mathfrak{I}(\Omega(G)) \subseteq \Omega(V)$.

Let $\Omega(G) = \langle c \mid c^2 = 1 \rangle \times \langle d \mid d^2 = 1 \rangle$. Then any $x \in \mathfrak{I}(\Omega(G))$ can be written as $x = \alpha(c+1) + \beta(d+1) + \gamma(c+1)(d+1)$, where $\alpha, \beta, \gamma \in KG$. It is easy to see that $x^2 = (\alpha\beta - \beta\alpha)(c+1)(d+1) = 0$ and $1 + \mathfrak{I}(\Omega(G)) \subseteq \Omega(V)$.

Lemma 4. *Let KG be the group algebra of an abelian 2-group G over a field K of characteristic 2. If $x \in KG$, then $x^2 = 0$ if and only if $x \in \mathfrak{I}(\Omega(G))$. Moreover,*

$$\Omega(V) = 1 + \mathfrak{I}(\Omega(G)).$$

□

Lemma 5. *Let $H = \langle c \rangle$ be cyclic of order 2^n and let $v^2 \in vKH(1 + c^{2^{n-1}})$ for some $v \in KH$. Then $v^2 = 0$.*

Proof. Let us write v as $\sum_{i=1}^{2^{n-1}-1} v_i c^i$ and $v_i \in K\langle c^{2^{n-1}} \rangle$. By induction on k ($2 \leq k \leq n$) we shall prove that

$$v(1 + c^{2^{n-1}}) = \sum_{i=0}^{2^{n-k}-1} \chi(v_i) c^i (1 + c^{2^{n-1}})(1 + c^{2^{n-2}}) \cdots (1 + c^{2^{n-k}}). \quad (4)$$

Clearly, $v_i^2 = \chi(v_i)^2$ and $v^2 = \sum_{i=1}^{2^{n-1}-1} \chi(v_i)^2 c^{2i}$. Since $v^2 \in vKH(1 + c^{2^{n-1}})$ we obtain that $\chi(v_i)^2 = \chi(v_{i+2^{n-2}})^2$ which implies that $\chi(v_i) = \chi(v_{i+2^{n-2}})$, where $i = 0, 1, \dots, 2^{n-2} - 1$. Therefore,

$$v(1 + c^{2^{n-1}}) = \sum_{i=0}^{2^{n-2}-1} \chi(v_i) c^i (1 + c^{2^{n-1}})(1 + c^{2^{n-2}})$$

and the equality (4) is true for $k = 2$. Assume that

$$v(1 + c^{2^{n-1}}) = \sum_{i=0}^{2^{n-k+1}-1} \chi(v_i) c^i (1 + c^{2^{n-1}})(1 + c^{2^{n-2}}) \cdots (1 + c^{2^{n-k+1}}). \quad (5)$$

Then $\chi(v_i) = \chi(v_{i+2^{n-k+1}}) = \chi(v_{i+2^{n-k+2}}) = \dots = \chi(v_{i+2^{n-2}})$, where $i = 0, \dots, 2^{n-1} - 2^{n-2} - 1$ and using this equality we get

$$v^2 = \sum_{i=1}^{2^{n-1}-1} \chi(v_i)^2 c^{2i} = \sum_{i=1}^{2^{n-k+1}-1} \chi(v_i)^2 c^{2i} (1 + c^{2^{n-1}})(1 + c^{2^{n-2}}) \cdots (1 + c^{2^{n-k+2}}). \quad (6)$$

Since $v^2 \in vKH(1 + c^{2^{n-1}})$, by (5) we have

$$v^2 \in KH(1 + c^{2^{n-1}})(1 + c^{2^{n-2}}) \cdots (1 + c^{2^{n-k+1}})$$

and by (6) $\chi(v_i) = \chi(v_{i+2^{n-k}})$ for $i = 1, 2, \dots, 2^{n-k} - 1$. It implies by (5) that (4) is proved for all k .

Let $k = n$. Then $v(1 + c^{2^{n-1}}) = \chi(v_0)(1 + c + c^2 + \dots + c^{2^{n-1}})$ and v^2 belongs to the ideal $K(1 + c + c^2 + \dots + c^{2^{n-1}})$. Clearly, $v^2 = \alpha(1 + c + c^2 + \dots + c^{2^{n-1}})$ for some $\alpha \in K$. Since $v^2 \in K\langle c^2 \rangle$ we conclude that $\alpha = 0$ and $v^2 = 0$. □

We remark that for any G from the Theorem, $G/\zeta(G)$ is elementary abelian. For any $x = \sum_{g \in G} \alpha_g g \in \Omega(V)$ we have $x^2 = \sum_{g \in G} \alpha_g^2 g^2 + w = 1$, where w belongs to

the commutator subspace $\mathcal{L}(KG)$. Obviously, $g^2 \in \zeta(G)$ and $\sum_{g \in G} \alpha_g^2 g^2 \in K\zeta(G)$. We know that $\mathcal{L}(KG) \cap K\zeta(G) = 0$. Therefore,

$$\sum_{g \in G} \alpha_g^2 g^2 = 1 \text{ and } w = 0. \quad (7)$$

We have the following cases:

Case 1. Let $G = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}} \rangle$ with $n, m \geq 2$. Any $x \in \Omega(V)$ can be written as $x = x_0 + x_1a + x_2b + x_3ab$, where $x_i \in K\zeta(G)$, and by (7) we have

$$x^2 = x_0^2 + x_1^2 a^2 + x_2^2 b^2 + x_3^2 a^{2+2^{n-1}} b^2 = 1$$

and $x_i^2 \in K\zeta^2(G)$. Since $\zeta(G) = \zeta^2(G) \cup a^2 \zeta^2(G) \cup b^2 \zeta^2(G) \cup a^2 b^2 \zeta^2(G)$ we have $x_0^2 = 1, x_1^2 = x_2^2 = x_3^2 = 0$ and by Lemma 4 we conclude that $\Omega(V) = 1 + \mathfrak{I}(\Omega(G))$.

Case 2. Now let $G = \langle a, b \mid a^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle \times \langle c \mid c^{2^n} = 1 \rangle$ with $n \geq 0$. Any $x \in \Omega(V)$ can be written as

$$x = x_0 + x_1a + x_2b + x_3ab,$$

where $x_i \in K\zeta(G)$. By (7) we have $x^2 = x_0^2 + (x_1^2 + x_2^2 + x_3^2)a^2 = 1$ and

$$x_0^2 = 1, \quad x_1^2 + x_2^2 + x_3^2 = 0, \quad x_i x_j (1 + a^2) = 0,$$

where $i, j = 1, 2, 3$ and $i \neq j$. Then $\chi(x_0) = 1$ and $\chi(x_1) + \chi(x_2) + \chi(x_3) = 0$.

We shall prove that $x_1^2 = x_2^2 = x_3^2 = 0$. Clearly, $\zeta(G) = \langle a^2 \rangle \times \langle c \rangle$ and $y^2 \in K\langle c^2 \rangle$, for any $y \in K\zeta(G)$. Since $x_i x_j (1 + a^2) = 0$ ($i \neq j$), we obtain that $x_i x_j = z_{ij}(1 + a^2)$, where $z_{ij} \in K\zeta(G)$. Clearly, by Lemma 4, $x_1 + x_2 + x_3 \in \mathfrak{I}(\Omega(\zeta(G)))$ and

$$x_1 + x_2 + x_3 = z_1(c^{2^{n-1}} + 1) + z_2(a^2 + 1),$$

where $z_1, z_2 \in K\langle c \rangle$. Thus

$$x_1^2 + x_1 x_2 + x_1 x_3 = z_1 x_1 (c^{2^{n-1}} + 1) + z_2 x_1 (a^2 + 1)$$

and $x_1^2 \equiv z_1 x_1 (c^{2^{n-1}} + 1) \pmod{\mathfrak{I}(\langle a^2 \rangle)}$, where $\mathfrak{I}(\langle a^2 \rangle) = K\zeta(G)(1 + a^2)$.

Since $K\zeta(G)/\mathfrak{I}(\langle a^2 \rangle) \cong K\langle c \rangle$, by Lemma 5 $x_1^2 \equiv 0 \pmod{\mathfrak{I}(\langle a^2 \rangle)}$. Clearly $x_i^2 \in K\langle c^2 \rangle$ and $K\langle c^2 \rangle \cap \mathfrak{I}(\langle a^2 \rangle) = 0$. Thus, $x_1^2 = 0$ and similarly we conclude that $x_i^2 = 0$, where $i = 2, 3$.

Therefore, by Lemma 4 $\Omega(V) = 1 + \mathfrak{I}(\Omega(G))$.

Case 3. Let $G = \langle a, b, d \mid a^4 = d^{2^n} = 1, a^2 = b^2, a^b = a^{-1}, d^a = d^{1+2^{n-1}}, b^d = b \rangle$. Any $x \in \Omega(V)$ can be written as

$$x = x_0 + x_1a + x_2b + x_3ab + x_4d + x_5ad + x_6bd + x_7abd,$$

where $x_i \in K\zeta(G)$. By (7) we have

$$x^2 = x_0^2 + (x_1^2 + x_2^2 + x_3^2)a^2 + x_4^2 d^2 + (x_5^2 + x_7^2)a^2 d^{2+2^{n-1}} + x_6^2 d^2 a^2 = 1 \quad (8)$$

and

$$\begin{cases} (x_1x_3 + x_5x_7d^{2+2^{n-1}})(1+a^2) = 0; \\ (x_1x_5 + x_3x_7)(1+d^{2^{n-1}}) = 0; \\ (x_1x_7a^2 + x_3x_5)(1+a^2d^{2^{n-1}}) = 0. \end{cases} \quad (9)$$

It is easy to see that $KG/\mathfrak{J}(\langle d^{2^{n-1}} \rangle) \cong K(Q_8 \times C_{2^{n-1}})$. By case 2, $x-1 \in \mathfrak{J}(\langle a^2, d^{2^{n-2}} \rangle)$, and

$$(x_0 + x_4d) + (x_1 + x_5d)a + (x_2 + x_6d)b + (x_3 + x_7d)ab \equiv 1 \pmod{\mathfrak{J}(\langle a^2, d^{2^{n-2}} \rangle)}.$$

We obtain that

$$\chi(x_0) + \chi(x_4) = 1 \text{ and } \chi(x_i) + \chi(x_{i+4}) = 0 \quad (10)$$

for all $i = 1, 2, 3$.

First, suppose that $n = 2$. Then $d^4 = 1$ and from (8), (9) and (10) we have that either $\chi(x_i) = 0$ ($i = 1, \dots, 7$) or $\chi(x_1) = \chi(x_3) = \chi(x_5) = \chi(x_7)$ and $\chi(x_2) = 0$.

If $\chi(x_1) = \chi(x_3) = \chi(x_5) = \chi(x_7) \neq 0$ and $\chi(x_2) = 0$, then x_1, x_3, x_5, x_7 are units and $x_i^{-1} = \chi(x_1)^{-1}x_i$ ($i = 1, 3, 5, 7$). Then (9) implies

$$\begin{cases} (1 + \chi(x_1)^{-1}x_1x_3x_5x_7)(1+a^2) = 0; \\ (1 + \chi(x_1)^{-1}x_1x_3x_5x_7)(1+d^2) = 0; \\ ((1+a^2) + 1 + \chi(x_1)^{-1}x_1x_3x_5x_7)(1+a^2d^2) = 0. \end{cases} \quad (11)$$

Since $1+a^2d^2 = (1+a^2)(1+d^2) + (1+a^2) + (1+d^2)$, by the third equality of (11) $(1+a^2)(1+a^2d^2) = 0$, which is impossible.

If now $\chi(x_j) = 0$ ($j = 1, 2, 3, 5, 7$), then $1+x_0, x_j \in \mathfrak{J}(\Omega(G))$ and by Lemma 4 $x \in 1 + \mathfrak{J}(\Omega(G))$.

Let $n > 2$. Clearly, $x \in 1 + \mathfrak{J}(\langle a^2, d^{2^{n-2}} \rangle)$,

$$\mathfrak{J}(\langle a^2, d^{2^{n-2}} \rangle) = KG(a^2 + 1) + KG(1 + d^{2^{n-1}}) + KG(1 + d^{2^{n-2}})$$

and

$$x_i \equiv \sum_{j=0}^{2^{n-2}-1} \alpha_{ij}d^{2j} \pmod{\mathfrak{J}(\langle a^2, d^{2^{n-1}} \rangle)},$$

where $\alpha_{ij} \in K$. Since $x \in 1 + \mathfrak{J}(\langle a^2, d^{2^{n-2}} \rangle)$ we have $\alpha_{ij} = \alpha_{ij+2^{n-3}}$, where $i = 1, \dots, 7, j = 0, \dots, 2^{n-3}-1$. Thus x_i can be written as

$$x_i = u_i(1+a^2) + v_i(1+d^{2^{n-1}}) + t_i(1+d^{2^{n-2}}),$$

where $u_i, v_i \in K\zeta(G)$ and

$$t_i = \sum_{j=0}^{2^{n-3}-1} \alpha_{ij}d^{2j}. \quad (12)$$

By (8) $x_0^2 - 1 = x_1^2 + x_2^2 + x_3^2 = x_4^2 = (x_5^2 + x_7^2)d^{2^{n-1}} + x_6^2 = 0$ and by Lemma 4 $x_0 - 1, x_4 \in \mathfrak{J}(\langle a^2, d^{2^{n-1}} \rangle)$. Thus

$$t_4^2(1+d^{2^{n-1}}) = (t_1^2 + t_2^2 + t_3^2)(1+d^{2^{n-1}}) = (t_5^2 + t_6^2 + t_7^2)(1+d^{2^{n-1}}) = 0. \quad (13)$$

Obviously, $\text{Supp}(t_i^2) \subseteq \langle d^4 \rangle$ and by (13) we conclude that $t_4^2 = t_1^2 + t_2^2 + t_3^2 = t_5^2 + t_6^2 + t_7^2 = 0$. It follows that

$$t_4 = t_1 + t_2 + t_3 = t_5 + t_6 + t_7 = 0. \quad (14)$$

We consider the projection of the equality $x^2 = 1$ onto cosets $a\zeta(G)$, $b\zeta(G)$, $ab\zeta(G)$, $d\zeta(G)$, $ad\zeta(G)$, $bd\zeta(G)$, $abd\zeta(G)$ of $\zeta(G)$ in G . Direct calculations show

$$\left\{ \begin{array}{l} (v_2t_3 + v_3t_2 + (u_4t_5 + (u_6 + v_6)t_7 + (u_7 + v_7)t_6)d^2) \times \\ \times (1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) + (t_2t_3 + t_6t_7d^2)(1 + a^2)(1 + d^{2^{n-1}}) = 0; \\ (v_1t_3 + t_1v_3 + (v_5t_7 + v_7t_5)d^2) \times \\ \times (1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) + (t_1t_3 + t_5t_7d^2)(1 + a^2)(1 + d^{2^{n-1}}) = 0; \\ (v_1t_2 + v_2t_1 + (u_4t_7 + (u_5 + v_5)t_6 + (u_6 + v_6)t_5)d^2) \times \\ \times (1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) + (t_1t_2 + t_5t_6d^2)(1 + a^2)(1 + d^{2^{n-1}}) = 0; \\ (u_1t_5 + u_5t_1 + u_3t_7 + u_7t_3)(1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) = 0; \\ (u_4t_1 + v_2t_7 + v_7t_2 + (u_3 + v_3)t_6 + (u_6 + v_6)t_3) \times \\ \times (1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) + (t_2t_7 + t_3t_6)(1 + a^2)(1 + d^{2^{n-1}}) = 0; \\ ((u_1 + v_1)t_7 + (v_7 + u_7)t_1 + (u_3 + v_3)t_5 + (u_5 + v_5)t_3) \times \\ \times (1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) + (t_1t_7 + t_3t_5)(1 + a^2)(1 + d^{2^{n-1}}) = 0; \\ ((v_1 + u_1)t_6 + (v_6 + u_6)t_1 + v_2t_5 + v_5t_2 + u_4t_3) \times \\ \times (1 + a^2)(1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) + (t_1t_6 + t_2t_5)(1 + a^2)(1 + d^{2^{n-1}}) = 0. \end{array} \right. \quad (15)$$

Put $\omega_0 = 1$, $\omega_k = \omega_{k-1}(1 + d^{2^{n-k-2}})$ ($k > 0$) and $t_i^{(k)} = \sum_{j=0}^{2^{n-k-3}} \alpha_{ij} d^{2j}$, where $k \geq 0$ and the α_{ij} are the same as in (12).

We shall establish by induction on k that $t_i = t_i^{(k)}\omega_k$. Clearly, $t_i^{(0)} = t_i$. Using the definition of ω_k we have

$$(1 + d^{2^{n-1}})(1 + d^{2^{n-2}})\omega_{k-1} = (1 + d^{2^{n-1}})(1 + d^{2^{n-2}}) \cdots (1 + d^{2^{n-k-1}}).$$

By the induction hypothesis the first three equations of (15) imply

$$\left\{ \begin{array}{l} (t_2^{(k-1)}t_3^{(k-1)} + t_6^{(k-1)}t_7^{(k-1)}d^2) \quad (1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}) = 0; \\ (t_1^{(k-1)}t_3^{(k-1)} + t_5^{(k-1)}t_7^{(k-1)}d^2) \quad (1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}) = 0; \\ (t_1^{(k-1)}t_2^{(k-1)} + t_5^{(k-1)}t_6^{(k-1)}d^2) \quad (1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}) = 0. \end{array} \right. \quad (16)$$

Because of (14) we can write

$$(t_1^{(k-1)} + t_2^{(k-1)} + t_3^{(k-1)})\omega_{k-1} = (t_5^{(k-1)} + t_6^{(k-1)} + t_7^{(k-1)})\omega_{k-1} = 0,$$

from which we obtain $t_1^{(k-1)} + t_2^{(k-1)} + t_3^{(k-1)} = t_5^{(k-1)} + t_6^{(k-1)} + t_7^{(k-1)}$. Using this equality we can put the expression of $t_3^{(k-1)}$ and $t_7^{(k-1)}$ into (16), and obtain

$$\left\{ \begin{array}{l} (t_2^{(k-1)} + t_6^{(k-1)}d^2) \quad (1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}) = 0; \\ (t_1^{(k-1)} + t_5^{(k-1)}d^2) \quad (1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}) = 0. \end{array} \right. \quad (17)$$

The supports of the elements $(t_i^{(k-1)})^2$ and $d^2(t_j^{(k-1)})^2$ belong to different cosets of $\langle d^4 \rangle$ in $\langle d^2 \rangle$. Thus, by the equality

$$(t_i^{(k-1)})^2(1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}) = \sum_{j=0}^{2^{n-k-3}-1} (\alpha_{ij} + \alpha_{ij+2^{n-k-3}}) d^{4j} (1 + d^{2^{n-1}}) \cdots (1 + d^{2^{n-k-1}}),$$

it follows from (17) that $\alpha_{ij} = \alpha_{ij+2^{n-k-3}}$ for all $i = 1, 2, 5, 6$ and $j = 0, \dots, 2^{n-k-3} - 1$. Hence,

$$t_i^{(k-1)} = \left(\sum_{j=0}^{2^{n-k-3}-1} \alpha_{ij} d^{2j} \right) (1 + d^{2^{n-k-2}}) = t_i^{(k)} (1 + d^{2^{n-k-2}}),$$

and since by induction $t_i = t_i^{(k-1)} \omega_{k-1}$, we conclude that $t_i = t_i^{(k)} \omega_k$, as desired.

By putting $k = n-3$ we get $t_i = \alpha_{i0}(1+d^2) \cdots (1+d^{2^{n-3}})$. Then taking $k = n-3$ in (17) we obtain that $t_i = t_{i+4} = \alpha_i(1+d^2) \cdots (1+d^{2^{n-3}})$, where $\alpha_i = \alpha_{i0}$ and $i = 1, 2, 3$.

Then from (15) we get the following system

$$\begin{cases} \alpha_1 \chi(u_4 + u_6 + v_2 + v_6) + \alpha_2 \chi(u_6 + u_7 + v_2 + v_3 + v_6 + v_7) + \alpha_2^2 + \alpha_1 \alpha_2 = 0; \\ \alpha_1 \chi(v_1 + v_3 + v_5 + v_7) + \alpha_2 \chi(v_1 + v_5) + \alpha_1^2 + \alpha_1 \alpha_2 = 0; \\ \alpha_1 \chi(u_4 + u_6 + v_2 + v_6) + \alpha_2 \chi(u_4 + u_5 + v_1 + v_5) + \alpha_1 \alpha_2 = 0; \\ \alpha_1 \chi(u_1 + u_3 + u_5 + u_7) + \alpha_2 \chi(u_3 + u_7) = 0; \\ \alpha_1 \chi(u_4 + u_6 + v_2 + v_6) + \alpha_2 \chi(u_3 + u_6 + v_2 + v_3 + v_6 + v_7) = 0; \\ \alpha_1 \chi(u_1 + u_3 + u_5 + u_7 + v_1 + v_3 + v_5 + v_7) + \alpha_2 \chi(u_1 + u_5 + v_1 + v_5) = 0; \\ \alpha_1 \chi(u_4 + u_6 + v_2 + v_6) + \alpha_2 \chi(u_1 + u_4 + v_1 + v_5) = 0. \end{cases}$$

It is easy to verify that the last system yields $\alpha_1 = \alpha_2 = 0$. Therefore, $x \in 1 + \mathfrak{I}(\Omega(G))$.

Case 4. Let

$$G = \langle x, y, u \mid x^4 = y^4 = 1, x^2 = [y, x], \\ y^2 = u^2 = [u, x], x^2 y^2 = [u, y] \rangle.$$

It is easy to see that $\zeta(G) = \langle x^2, y^2 \rangle$. Then any $\alpha \in \Omega(V)$ can be written as

$$\alpha = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy + \alpha_4 u + \alpha_5 xu + \alpha_6 yu + \alpha_7 xyu,$$

where $\alpha_i \in K\zeta(G)$. Clearly $\alpha_i^2 \in K$ and by (7) we have

$$\alpha^2 = \alpha_0^2 + (\alpha_1^2 + \alpha_5^2 + \alpha_7^2)x^2 + (\alpha_2^2 + \alpha_3^2 + \alpha_4^2)y^2 + \alpha_6^2 x^2 y^2 = 1 \quad (18)$$

and

$$\begin{cases} \alpha_1 \alpha_2 (1 + x^2) + \alpha_4 \alpha_7 (1 + x^2) y^2 = 0; \\ \alpha_1 \alpha_3 (1 + x^2) + \alpha_4 \alpha_6 (x^2 + y^2) + \alpha_5 \alpha_7 (1 + y^2) = 0; \\ \alpha_1 \alpha_4 (1 + y^2) + \alpha_2 \alpha_7 (1 + y^2) x^2 = 0; \\ \alpha_1 \alpha_5 (1 + y^2) x^2 + \alpha_2 \alpha_6 (x^2 + y^2) + \alpha_3 \alpha_7 (1 + x^2) y^2 = 0; \\ \alpha_1 \alpha_6 (1 + x^2 y^2) + \alpha_2 \alpha_5 (1 + y^2) x^2 + \alpha_3 \alpha_4 (1 + x^2) = 0; \\ \alpha_1 \alpha_7 (x^2 + y^2) + \alpha_2 \alpha_4 (1 + x^2 y^2) = 0; \\ \alpha_2 \alpha_3 (1 + x^2) y^2 + \alpha_4 \alpha_5 (1 + y^2) + \alpha_6 \alpha_7 (1 + x^2 y^2) = 0. \end{cases} \quad (19)$$

From (18) we have $\alpha_1^2 + \alpha_5^2 + \alpha_7^2 = \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = \alpha_6^2 = 0$ and $\alpha_0^2 = 1$. Therefore, $\chi(\alpha_5) = \chi(\alpha_1) + \chi(\alpha_7)$ and $\chi(\alpha_3) = \chi(\alpha_2) + \chi(\alpha_4)$. If we multiply (19) first by $1+x^2$ and then by $1+y^2$ and replace $\chi(\alpha_5)$ by $\chi(\alpha_1) + \chi(\alpha_7)$ and $\chi(\alpha_3)$ by $\chi(\alpha_2) + \chi(\alpha_4)$ we obtain $\chi(\alpha_j) = 0$ for all $j = 1, \dots, 7$.

Therefore, $\Omega(V) = 1 + \mathfrak{I}(\Omega(G))$, which completes the proof of the Theorem. \square

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